Lecture 5

The one with a lot of σ.

Lemma. Let X, Y be manifolds with transitive smooth actions of G. Then any G-equivariant map $X \xrightarrow{f} Y$ is a locally trivial fiber bundle, and the typical fiber is a homogeneous space of a closed subgroup of G. (Don't need to assume f smooth!)

Pf sketch. Let $f(x_0) = y_0$ and identify $X \cong G/H$, $Y \cong G/K$ with x_0 corresp to eH and y_0 to eK. So $H = \{g \mid g x_0 = x_0\}$ etc.

Then G-equivariance gives $g \cdot x_0 = x_0 \Rightarrow g \cdot y_0 = g \cdot f(x_0) = f(g \cdot x_0)$
$= f(x_0) = y_0$.

So $g \in H \Rightarrow g \in K$, $H \subset K$.

Thus there's a natural map $G/H \to G/K$, $gH \xmapsto{\mu} gK$. (equivt)

Claim $\mu = f$: $f(gH) = f(g \cdot eH) = g \cdot f(eH) = g \cdot (eK) = gK = \mu(gH)$

So all we need is $G/H \to G/K$ is a locally trivial fibration with typical fiber K/H.

$G \to G/K$ admits local sections (submersions always do)

$$\begin{tikzcd} G \arrow[d, "\pi_K"'] & \supset & S \\ G/K & \supset & \mathcal{U} \arrow[u, "\sigma"'] \end{tikzcd}$$

$\mathcal{U} \times K \xrightarrow{\alpha} G$ is an embedding
$(x, k) \longmapsto \sigma(x) \cdot k$

this implication requires equivariance

Equivariant for right mult by H (or K) so it induces a smooth injective

$\mathcal{U} \times K/H \xrightarrow{\bar{\alpha}} G/H$ which can be shown to be embedding (constant rank)

This is the local triv:

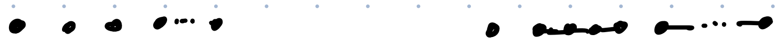
$$\begin{tikzcd} \mathcal{U} \times K/H \arrow[r, "\bar{\alpha}"] \arrow[d] & G/H \arrow[d, "\mu"] \\ \mathcal{U} & \subset \; G/K \end{tikzcd}$$

$\square$

Cor. Any map that just omits some parts of a flag is a surjective equivariant locally triv fibration with fibers that are products of flag varieties.

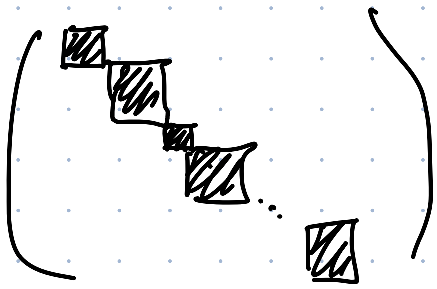
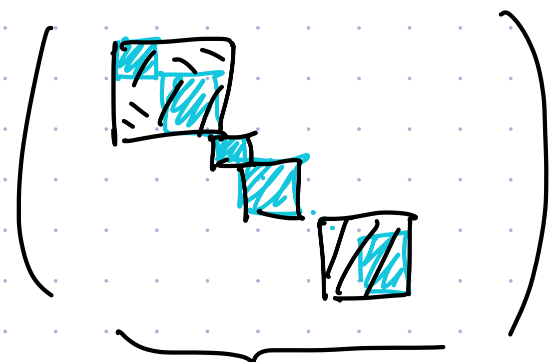
e.g. $(F_1 \subset F_2 \subset \cdots \subset F_n) \longmapsto (F_1 \subset F_5 \subset F_n)$

$\quad 1, \ldots, 1 \qquad\qquad\qquad 1, 4, n-5$



Pf. Clearly equivt. Hence smooth. Local triviality by above.

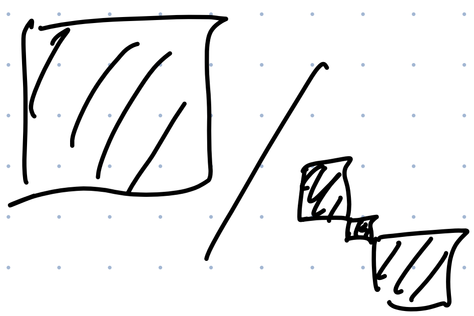
In the $O(n)$ or $U(n)$ model, H and K look like

$H =$  $K =$ 

Each block is orthog / unitary

some blocks from H merge into a bigger block. (merge $\Leftrightarrow$ the subspace between is omitted)

So the quotient is a product of things like



i.e. $O(k) / \prod O(k_i)$ $\quad \sum k_i = k$.

i.e. $\mathrm{Flag}(W, \underline{k})$

W: k-dim

Ex. *$\mathrm{Flag}(V, 1, n-2, 1) = \{(\ell, H)\}$ $\quad \ell$ line $\quad H$ hyperplane $\quad \ell \subset H$

$$\begin{array}{ccc} \mathrm{Flag}(V,1,n-2,1) & \longrightarrow & \mathrm{Flag}(V,1,n-1) \\ (\ell, H) & \longmapsto & H \end{array}$$

is loc triv, fiber over H is $\mathbb{P}(H)$

*This is called the incidence variety

$$\begin{array}{ccc} Flag(V,1,n-2,1) & \longrightarrow & Flag(V,n-1,1) \\ (\ell, H) & \longmapsto & \ell \end{array}$$ is loc triv, fibers over ℓ is $\mathbb{P}^*(V/\ell)$.

Complex simple and semisimple Lie alg

G Lie group $\mathfrak{g} = \mathrm{Lie}(G)$ Lie alg $= \{$ Left invt vec fields on $G \}$ w/ Lie bracket $\underset{vec}{\cong} T_e G$.

a Lie algebra over $\mathbb{R}$

complex Lie group

$GL_n\mathbb{C}$ = invertible $\rightsquigarrow$ $\mathfrak{gl}_n\mathbb{C}$ all $n\times n$ matrices (matrix comm)

Lie alg/$\mathbb{C}$

$SL_n\mathbb{C} = \{\det = 1\} \longrightarrow \mathfrak{sl}_n\mathbb{C}$ traceless ($\mathrm{tr}\, A = 0$)

$\mathfrak{sl}_n\mathbb{C}$ is an example of a complex simple Lie alg

$\mathfrak{sl}_n\mathbb{C} \oplus \mathfrak{sl}_k\mathbb{C}$ is an example of a complex semi-simple Lie alg.

If this is an example, what else is there? Simple ones admit a beautiful classification. (Due to E. Cartan 1894, W. Killing 1890)

[Knapp: Lie Groups Beyond an Introduction, I.5–I.8, II.1–II.5]

[Laksh-Brown Chap 7]. $\mathfrak{g}$ Lie alg. $\mathfrak{a} < \mathfrak{g}$ subalg.

$\mathfrak{a}$ is called an *ideal* if $[\mathfrak{g}, \mathfrak{a}] < \mathfrak{a}$. ($\approx$ normal subgp of G)

A Lie algebra is *solvable* if it is inductively constructed from abelian Lie alg, specifically:

Define $D^0\mathfrak{g} = \mathfrak{g}$ and $D^k\mathfrak{g} = [D^{k-1}\mathfrak{g}, D^{k-1}\mathfrak{g}]$ The derived series of $\mathfrak{g}$

Then solvable $:=$ $D^k\mathfrak{g} = \{0\}$ for some k ($\Rightarrow D^{k-1}$ abelian)

Ex. Let $\mathfrak{g} = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}$. Then $[\mathfrak{g},\mathfrak{g}] = \begin{pmatrix} 0 & * & \cdots & * \\ & \ddots & \ddots & \vdots \\ & & \ddots & * \\ & & & 0 \end{pmatrix}$

$[D^1\mathfrak{g}, D^1\mathfrak{g}] = \begin{pmatrix} 0 & 0 & * \\ & \ddots & \ddots \\ & & 0 \\ & & & 0 \end{pmatrix}$ etc.

is solvable.

Cor. $\mathfrak{gl}_n\mathbb{C}$, $\mathfrak{sl}_n\mathbb{C}$, $\mathfrak{gl}_n\mathbb{R}$, $\mathfrak{sl}_n\mathbb{R}$ are not solvable (exercise!) but contain solvable subalgebras.

However, $\mathfrak{sl}_n\mathbb{C}$, $\mathfrak{sl}_n\mathbb{R}$ contain no solvable ideals. (No ideals at all, in fact!)

while $\mathfrak{a} = \mathbb{C}\cdot I_{n\times n} \subset \mathfrak{gl}_n\mathbb{C}$ is an abelian (hence solvable) ideal.

When $\mathfrak{a} \subset \mathfrak{g}$ is an ideal, we get a well-def bracket on $\mathfrak{g}/\mathfrak{a}$. So we have an exact sequence of Lie algebras

$$0 \to \mathfrak{a} \to \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{a} \to 0. \qquad (*)$$

Def. $\mathrm{rad}(\mathfrak{g})$ (radical) = the unique maximal solvable ideal of G.

Exists: if $\mathfrak{a}, \mathfrak{b}$ are solvable ideals, so is $\mathfrak{a}+\mathfrak{b}$. (ideal!)

$\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is a Lie alg w/ no solvable ideals except $\{0\}$.

So understanding Lie alg $\mathfrak{g}$ can be split into understanding those that are solvable, those that have no solvable ideals, and extensions

$$\mathrm{rad}(\mathfrak{g}) \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$$

Def. A Lie algebra is semisimple if $\mathrm{rad}(\mathfrak{g}) = 0$
$\iff$ every abelian ideal is trivial.

Def. A Lie algebra is simple if it has no nontrivial ideals.
(compare: simple group, normal subgps)

Thm: A Lie alg is semisimple iff it is a direct sum of simple Lie algebras.